IRREDUCIBLE DISCONNECTED SYSTEMS IN GROUPS

BY

GIL KAPLAN*

School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences Tel Aviv University, 69978 Tel Aviv, Israel
e-mail: kaplagi@math.tau.ac.il

ABSTRACT

Let G be an arbitrary group with a subgroup A. The **subdegrees** of (A, G) are the indices $[A : A \cap A^g]$ (where $g \in G$). Equivalent definitions of that concept are given in [IP] and [K]. If A is not normal in G and all the subdegrees of (A, G) are finite, we attach to (A, G) the **common divisor graph** Γ: its vertices are the non-unit subdegrees of (A, G), and two different subdegrees are joined by an edge iff they are *not* coprime. It is proved in [IP] that Γ has at most two connected components. Assume that Γ is disconnected. Let D denote the subdegree set of (A, G) and let D_1 be the set of all the subdegrees in the component of Γ containing $\min(D - \{1\})$. We proved [K, Theorem A] that if A is stable in G (a property which holds when A or $[G : A]$ is finite), then the set $H = \{g \in G | \, [A : A \cap A^g] \in D_1 \cup \{1\}\}$ is a subgroup of G. In this case we say that $A < H < G$ is a **disconnected system** (briefly: a **system**). In the current paper we deal with some fundamental types of systems. A system $A < H < G$ is **irreducible** if there does not exist $1 < N \lhd G$ such that $AN < H$ and $AN/N < H/N < G/N$ is a system. Theorem A gives restrictions on the finite nilpotent normal subgroups of G, when G possesses an irreducible system. In particular, if G is finite then $\mathrm{Fit}(G)$ is a q-group for a certain prime q. We deal also with general systems. Corollary (4.2) gives information about the structure of a finite group G which possesses a system. Theorem B says that for any system $A < H < G$, $N_G(N_G(A)) = N_G(A)$. Theorem C and Corollary C′ generalize a result of Praeger [P, Theorem 2].

---

* The content of this paper corresponds to a part of the author's Ph.D. thesis carried out at Tel Aviv University under the supervision of Prof. Marcel Herzog.

Received May 22, 1997 and in revised form October 5, 1997

1. Introduction

In this paper we deal with groups G possessing a subgroup A, such that the index set $D = \{[A : A \cap A^g] |\ g \in G\}$ satisfies the following property: it consists of finite numbers only, and there exist two disjoint and non-empty sets of primes, σ and τ, such that $D - \{1\}$ is a union of a non-empty set of σ-numbers and a non-empty set of τ-numbers. Isaacs and Praeger [IP] proved that for any subgroup A, there exist *at most two* disjoint sets of primes satisfying that condition.

We have shown in [K] that in this situation, provided that A is stable in G (a property which holds when A or $[G : A]$ is finite, and will be defined later in this paper), the group G has a nice structure. The main result in [K] ([K], Theorem A) is as follows: fix the notation for σ and τ such that $\min(D - \{1\})$ is a σ-number. Then *the set* $H = \{g \in G |\ [A : A \cap A^g]$ is a σ-number$\}$ *is a subgroup of* G *if and only if* A *is stable in* G (note that 1 is considered as a σ-number).

In the current paper we obtain further information about groups G possessing such a subgroup A, provided that A is stable in G. In that case we say that $A < H < G$ is a **disconnected system** (briefly: a **system**) of G. We say that a system $A < H < G$ is **irreducible** if there does not exist a system of the form $AN/N < H/N < G/N$, where $1 < N \lhd G,\ AN < H$.

It is easy to see that for any system $A < H < G$, A is not a finite nilpotent group (see the remark in section 4). In section 3 we deal with irreducible systems. One of our results is that if G is a finite group possessing an irreducible system, then $\mathrm{Fit}(G)$ is a q-group for a certain prime q (in this paper the notion "q-group"s includes trivial groups; $\mathrm{Fit}(G)$ denotes the Fitting subgroup of G). In section 4 general systems are considered. We prove that if G is a finite group possessing a system $A < H < G$, then G has a proper normal subgroup N, such that $\mathrm{Fit}(G/N)$ is a q-group for a certain prime q, and AN/N is not nilpotent. Another result is that if $A < H < G$ is a system then $N_G(N_G(A)) = N_G(A)$.

For a given pair (A, G) (where A is a subgroup of G), we continue to use the notation $D = \{[A : A \cap A^g] |\ g \in G\}$. An index $[A : A \cap A^g]$ is called a **subdegree** of (A, G), and so D is the **subdegree set** of (A, G). This concept was presented by Isaacs and Praeger ([IP]). In their paper the permutation group point of view is prominent: let G act transitively on a set X, and let $A = G_x$ be the stabilizer of a point $x \in X$. Then the subdegrees of (A, G) are exactly the cardinalities of the A-orbits on X.

In the remaining part of this section, A is a non-normal subgroup of G (equivalently: $D \neq \{1\}$), and we assume that all the subdegrees of (A, G) are finite. The **common divisor graph** Γ of (A, G) is the undirected graph with vertex set $D \setminus \{1\}$, in which two different vertices $s, t \in D - \{1\}$ are joined by an edge iff $\gcd(s,t) \neq 1$ ($\gcd(s,t)$ denotes the greatest common divisor of s and t). We shall occasionally write $\Gamma_{(A,G)}$, to emphasize the pair to which Γ corresponds. Isaacs and Praeger proved ([IP], Theorem A) that Γ has at most two connected components.

Remark: According to the definition in [IP], the common divisor graph includes also the trivial component of the vertex 1, and so, in terms of [IP], it has at most three components. ∎

In this paper, whenever Γ is disconnected, the vertex sets of the two components are denoted by D_1 and D_2. Furthermore, we always fix the notation such that $\min(D - \{1\}) \in D_1$.

We introduce now the important concepts of **pairing** and **stability**. The pairing concept played a significant role in [IP]. The related concept of stability was defined in [K]. Subdegrees s and t (not necessarily different) are **paired** if there exists $g \in G$ such that $s = [A : A \cap A^g]$, $t = [A : A \cap A^{g^{-1}}]$. Note that $[A : A \cap A^{g^{-1}}] = [A^g : A^g \cap A]$. Note further that a subdegree can be paired with more then one subdegree, and that every subdegree is paired with something. We say that A is **stable** in G if 1 is paired only with itself and every two paired subdegrees different from 1 lie in the same connected component of Γ. Let A_G be the core of A in G. It is easy to see that if A/A_G is finite then every subdegree is paired only with itself and so A is stable in G. In particular, if A or $[G : A]$ is finite then A is stable in G. Furthermore, (2.7) and (4.1)(a) in [IP] imply that if Γ is disconnected and D_1 or D_2 is a finite set, then A is stable in G.

Isaacs and Praeger gave detailed information about the subdegree set D in the disconnected and stable setting. Parallel information about the structure of the group G was presented in [K]. The main result there was ([K], Theorem A):

(1.1) THEOREM: *Assume that Γ is disconnected and denote*

$$H = \{g \in G \mid [A : A \cap A^g] \in D_1 \cup \{1\}\}.$$

Then H is a subgroup of G iff A is stable in G. In any case, $N_G(A) \subset H \subset G$, where both inclusions are proper.

Let $M \leq G$ and let $S \subseteq G$ be a non-empty union of right cosets Mu. Then we denote $S/M = \{Mu \mid u \in G,\ Mu \subseteq S\}$. Further we denote the cardinality of S/M by $[S : M]$. In particular $[AgA : A]$ is the cardinality of the set $\{Au \mid u \in G,\ Au \subseteq AgA\}$. The equality $[A : A \cap A^g] = [AgA : A]$ is easily proved (see [K], section 1). Other useful equalities are as follows: let M and S be as described above, and let $g, h \in G$. Then $[S^g : M^g] = [S : M]$, $[Sh : M] = [S : M]$, $[g^{-1}Sh : M^g] = [S : M]$. For a subgroup $M \leq G$, we denote by $M \setminus G$ the set of all the right cosets Mg ($g \in G$). The **standard action** of G on $M \setminus G$ is the transitive action defined by $(Mg)^u = Mgu$ (for $u \in G$). The core of M in G is denoted by M_G. The normal closure of M in G is denoted by M^G.

Theorem (1.1) motivates the following definitions. Let G be a group with a subgroup A and a subset H such that $A \subset H \subset G$. We say that $A \subset H \subset G$ is a **disconnected semi-system** (briefly: a **semi-system**) if A is not normal in G, all the subdegrees of (A, G) are finite, $\Gamma_{(A,G)}$ is disconnected and $H = \{g \in G | [A : A \cap A^g] \in D_1 \cup \{1\}\}$. Furthermore, if in addition A is stable in G then, by (1.1), H is a subgroup of G, and we say that $A < H < G$ is a **disconnected system** (briefly: a **system**). Notice that by (1.1) a semi-system $A \subset H \subset G$ is a system *if and only if* H is a subgroup of G.

A system $A < H < G$ is **reducible** if there exists N such that $1 < N \triangleleft G$, $AN < H$ and $AN/N < H/N < G/N$ is a system. If such N fails to exist, we say that the system is **irreducible**. Notice that if $A < H < G$ is irreducible, then the core A_G must be trivial.

Let a group G satisfy the maximal condition on normal subgroups (i.e., G does not have an infinite properly ascending chain of normal subgroups), and let $A < H < G$ be a system. By the definitions, it follows easily that there exists N such that $N \triangleleft G$, $AN < H$ and $AN/N < H/N < G/N$ is an irreducible system. Thus the concept of irreducible systems is fundamental in the research of systems in general.

In section 3 we prove the following criterion for irreducibility.

(3.3) PROPOSITION: *Let a group G possess a system $A < H < G$. Let $L = A^H$, the normal closure of A in H. Then $A < H < G$ is irreducible iff $AN \geq L$ for each N such that $1 < N \triangleleft G$.*

For a natural number n we denote by $\pi(n)$ the set of all the prime divisors of n. For a finite subgroup $M \leq G$ we write $\pi(M) = \pi(|M|)$. For a finite G and a set of primes σ, $O_\sigma(G)$ denotes the maximal normal σ-subgroup of G. Let $A < H < G$ be a system and let $L = A^H$. Then $[L : A]$ is finite and $[L : A] \neq 1$ by Theorem B of [K]. By using Proposition (3.3), we obtain in section 3 the following result about groups possessing an irreducible system.

THEOREM A: *Let a group G possess an irreducible system $A < H < G$. Let $L = A^H$. Suppose that N is a nontrivial finite normal subgroup of G. Then*

(i) *$\pi([L : A]) \subseteq \pi(N)$, and in particular*
(ii) *if N is nilpotent then $[L : A]$ is a power of a prime q, and N is a q-group.*

Theorem A immediately implies

COROLLARY A′: *Let a finite group G possess an irreducible system $A < H < G$. Let $L = A^H$. Then*
(i) *For each prime q dividing $[L : A]$, $O_{q'}(G) = 1$.*
(ii) *If $\mathrm{Fit}(G) \neq 1$, then $[L : A]$ is a power of a prime q and $\mathrm{Fit}(G)$ is a q-group.*

COROLLARY A″: *Let a finite solvable group G possess an irreducible system $A < H < G$. Let $L = A^H$. Then $[L : A]$ is a power of a prime q, $[H : N_G(A)]$ divides $[L : A]$, and $\mathrm{Fit}(G)$ is a q-group.*

Proof: Since G is solvable, $\mathrm{Fit}(G) \neq 1$. In view of Corollary A′, it is left to show only that $[H : N_G(A)]$ divides $[L : A]$. Since $H = LN_G(A)$ by Lemma (2.3) in [K], we have $[H : N_G(A)] = [L : L \cap N_G(A)]$. The result follows. ∎

In section 4 general systems are considered. One of the results is

THEOREM B: *Let a group G possess a system $A < H < G$. Then $N_G(N_G(A)) = N_G(A)$.*

In section 5 we deal mainly with systems $A < H < G$ satisfying $[L : A] = q$, a prime. This subfamily (alongside the subfamily of irreducible systems) seems as a good starting point for the research of systems in general. Our results, Theorem C and Corollary C′ below, generalize the following result of Praeger ([P], Theorem 2).

THE $\{1,p,q\}$-THEOREM: *Assume that $D = \{1,p,q\}$, where p and q are primes, $p < q$. Then p divides $q-1$, and A is contained in a normal subgroup N of G, such that $[N : A] = q$ and N/A_N is Frobenius of order pq.*

Notice that under the conditions of the $\{1,p,q\}$-Theorem, the sets $D_1 = \{p\}$ and $D_2 = \{q\}$ are finite, and so A must be stable in G. Thus a system $A < H < G$ occurs. Let $L = A^H$, then $[L : A]$ divides $\gcd(D_2)$ by Theorem B(ii) in [K] (where $\gcd(D_2)$ is the greatest common divisor of all the numbers in D_2). Hence $[L : A] = q$. In fact, we shall show (see the remark in section 5) that the normal subgroup N mentioned in the $\{1,p,q\}$-Theorem is equal to L. Hence the following results are indeed generalizations of the $\{1,p,q\}$-Theorem.

THEOREM C: *Let a group G possess a system $A < H < G$. Let $L = A^H$, and suppose that $[L : A] = q$, a prime. Then one of the following cases holds:*

(a) *L/A_L is Frobenius with a kernel of order q and a complement A/A_L of order, say, r. In this case $D_1 = \{r\}$.*

(b) *L/A_L is isomorphic to a nonsolvable 2-transitive permutation group of degree q. In this case $q - 1 \in D_1$.*

Furthermore, in both cases ((a) and (b)) if A is finite then A_L contains all the Sylow q-subgroups of A, which are nontrivial.

Remark: Notice again that $[L : A]$ divides $\gcd(D_2)$ by Theorem B(ii) in [K]. Thus the assumption $[L : A] = q$ of Theorem C holds in particular if $\gcd(D_2) = q$, a prime.

The case when $D_2 = \{q\}$ is of particular interest. In this case A must be stable in G (since D_2 is finite), and a system $A < H < G$ occurs. We have:

COROLLARY C′: *Assume that Γ is disconnected and $D_2 = \{q\}$, where q is a prime. Let $A < H < G$ be the corresponding system, and let $L = A^H$. Then $[L : A] = q$, and so one of the cases* (a) *and* (b) *of* Theorem C *holds. Moreover, $L = A^G$, the normal closure of A in G.*

ACKNOWLEDGEMENT: The author is grateful to the referee for his constructive remarks.

2. Notation

Throughout this paper, whenever $A < H < G$ is a system, we set the following notation:

A is a stable subgroup of G;

Γ (or $\Gamma_{(A,G)}$) is the common divisor graph of (A, G);

D is the subdegree set of (A, G);

D_1 is the set of all the subdegrees in the connected component of Γ containing $\min(D - \{1\})$;

D_2 is the set of all the subdegrees in the other component of Γ;

$H = \{g \in G | \, [A : A \cap A^g] \in D_1 \cup \{1\}\}$;

$L = A^H$, the normal closure of A in H.

3. Irreducible systems

We begin with the following two lemmas, which absorb the main part of the proof of Theorem A.

(3.1) LEMMA: *Let $A < G$, $N \lhd G$, $g \in G$, and suppose that $[A : A \cap A^g]$ is finite. Then $[AN : AN \cap (AN)^g]$ is finite, and it divides $[A : A \cap A^g]$.*

Proof: We have $[AN : AN \cap (AN)^g] = [ANgAN : AN] = [ANgA : AN] = [(AN)^g A : (AN)^g] = [A : A \cap (AN)^g]$. Since $[A : A \cap (AN)^g]$ divides $[A : A \cap A^g]$, the proof is concluded. ∎

(3.2) LEMMA: *Let a group G possess a system $A < H < G$, and let $N \lhd G$, $AN \not\geq L$. Then $AN < H$ and $AN/N < H/N < G/N$ is a system.*

Proof: Since $AN \not\geq L$, Theorem B(iii) of [K] implies $AN < H$. For each element $g \in G$ and subgroup $M \leq G$ denote $g^* = gN$, an element of G/N, and $M^* = MN/N$, a subgroup of G/N. We must prove that $A^* < H^* < G^*$ is a system. For $g \in G$ we have $[A^* : A^* \cap (A^*)^{g^*}] = [AN : AN \cap (AN)^g]$, and this last index divides $[A : A \cap A^g]$ by (3.1). Now let $u \in H$, $v \in G - H$, then $[A^* : A^* \cap (A^*)^{u^*}]$ divides $[A : A \cap A^u]$ and $[A^* : A^* \cap (A^*)^{v^*}]$ divides $[A : A \cap A^v]$. Thus $[A^* : A^* \cap (A^*)^{u^*}]$ and $[A^* : A^* \cap (A^*)^{v^*}]$ are coprime.

Let us check which elements $g \in G$ satisfy $[A^* : A^* \cap (A^*)^{g^*}] = 1$. For such g it holds that $AN \leq (AN)^g$. Hence $(AN)^{g^{-1}} \leq AN$, i.e. $A^{g^{-1}}N \leq AN$, and so both A and $A^{g^{-1}}$ are contained in AN. If $g \in G \setminus H$ then $AA^{g^{-1}} = LL^{g^{-1}} \supseteq L$ by Theorem B(i) in [K], whence $AN \geq L$, a contradiction. Thus $[A^* : A^* \cap (A^*)^{g^*}] = 1$ implies $g \in H$ and $g^* \in H^*$. Now if *each* $g^* \in H^*$ satisfies $[A^* : A^* \cap (A^*)^{g^*}] = 1$, then $A^* \lhd H^*$, whence $AN \lhd H$. Thus $AN \geq A^H = L$, a contradiction. It follows that $\{g^* \in G^* | \, [A^* : A^* \cap (A^*)^{g^*}] = 1\} \subset H^*$ (a *proper* inclusion).

From what we have shown till now we deduce that Γ^*, the common divisor graph of (A^*, G^*), is disconnected. By Theorem A of [IP] Γ^* has exactly two components. The vertex sets of these components must be

$$S = \{[A^* : A^* \cap (A^*)^{u^*}] | \, u^* \in H^*\} - \{1\} \quad \text{and}$$
$$T = \{[A^* : A^* \cap (A^*)^{v^*}] | \, v^* \in G^* - H^*\}.$$

For completing the proof that $A^* < H^* < G^*$ is a system, it suffices to show that the minimal non-unit subdegree of (A^*, G^*) is contained in S. We prove first that A^* is stable in G^*. Notice that 1 is paired only with itself, since Γ^* is disconnected (see [IP], (2.7)). Furthermore, let $g^* \in G^*$, then $[A^* : A^* \cap (A^*)^{g^*}] \in T$ iff

$g^* \in G^* - H^*$ iff $(g^*)^{-1} \in G^* - H^*$ iff $[A^* : A^* \cap (A^*)^{(g^*)^{-1}}] \in T$. Thus A^* is indeed stable in G^*.

Suppose now that the minimal non-unit subdegree is contained in T. Then the set $\{g^* \in G^* | \ [A^* : A^* \cap (A^*)^{g^*}] \in T \cup \{1\}\}$ is a proper subgroup of G^*, by the "if" part of (1.1). Denote this subgroup by K^*, then we have $G^* = H^* \cup K^*$, which implies $H^* = G^*$ and $H = G$, a contradiction. Thus the minimal non-unit subdegree is contained in S, and so the proof of the lemma is concluded. ∎

Lemma (3.2) enables us to obtain the following simple criterion for irreducibility.

(3.3) PROPOSITION: *Let a group G possess a system $A < H < G$. Then $A < H < G$ is irreducible iff $AN \geq L$ for each N such that $1 < N \lhd G$.*

Proof: Suppose first that there exists N such that $1 < N \lhd G$ and $AN \not\geq L$. Then $AN/N < H/N < G/N$ is a system by (3.2), whence $A < H < G$ is reducible. Next, suppose that $A < H < G$ is reducible. Thus there exists N such that $1 < N \lhd G$, $AN < H$, and $AN/N < H/N < G/N$ is a system. The proof will be concluded by showing that $AN \not\geq L$. Indeed, if $AN \geq L$ then $AN = LN$ and so $AN \lhd H$. Thus $AN/N \lhd H/N$. This is a contradiction, since $N_{G/N}(AN/N) < H/N$ by Theorem (1.1). ∎

Proposition (3.3) plays a central role in

Proof of Theorem A: (i) Since the system $A < H < G$ is irreducible, (3.3) implies $AN \geq L$. Thus $[L : A]$ divides $[AN : A]$, and so $[L : A]$ divides $[N : N \cap A]$. Hence $\pi([L : A]) \subseteq \pi(N)$.

(ii) This follows immediately by (i), since if N is nilpotent then each of its Sylow subgroups is normal in G. ∎

In the remaining part of this section we consider some examples for groups possessing an irreducible system.

(3.4) Example: It is easily verified that the "elementary abelian by Frobenius family" ([K], section 6) consists of finite solvable groups G possessing an irreducible system $A < H < G$, such that $[L : A] = [H : A] = q$, a prime. By Corollary A$''$ Fit(G) must be a q-group. In fact, here Fit(G) is a Sylow q-subgroup of G.

(3.5) Example: We present irreducible systems in which $[L : A] = q$, a prime, but Fit$(G) = 1$. Let q be a prime, $q \geq 5$, let $K = S_q$ (the symmetric group on $\{1, 2, \ldots, q\}$) and let $M = \{g \in K | \ g$ fixes the point $q\}$. Evidently $[K : M] = q$,

and since K is 2-transitive, the subdegree set of (M, K) is $\{1, q-1\}$. Choose a finite nontrivial q'-group R. We describe a subfamily of the "wreath product family" ([K], section 3). Consider the group G, which is the wreath product of K by R with respect to the action of R on itself by right multiplication (i.e., the standard action of R on the cosets of the trivial subgroup): $G = K \text{ wr } R = (K \times K \times \cdots \times K)R$. Each of the $|R|$ copies of K is related to one of the elements of R, and we agree that the first copy is related to the element 1 (this last convention is just to secure consistency with the notation of section 3 in [K]). Let $A = M \times K \times \cdots \times K < G$ ($|R| - 1$ copies of K); then the subdegree set of (A, G) is $\{1, q-1\} \cup \{q\} = \{1, q-1, q\}$, and so $\Gamma_{(A,G)}$ is disconnected (for details see [K], section 3). We have $H = K \times \cdots \times K$ ($|R|$ copies of K).

Suppose that there exists N such that $1 < N \lhd G$ and $AN < H$. Denote $J = \{g \in K|$ there exists in N an element of the form $(g, \ldots)\}$. Since $N \lhd G$ we have $J \trianglelefteq K$ and so $J = 1$ or $J = K$ or $J = K_0$, the alternating group on $\{1, 2, \ldots, q\}$. If $J = K$ or $J = K_0$ then $MJ = K$ and so $AN = H$, a contradiction. Thus $J = 1$. Since the action of R is transitive, the normality of N in G implies $N = 1$, a contradiction. Hence such N does not exist and the system $A < H < G$ is irreducible. Notice also that $L = H$, whence $[L : A] = [K : M] = q$. However, here $\text{Fit}(G) = 1$. Indeed, By Corollary A$'$(ii), $\text{Fit}(G)$ must be a q-group. Since we chose R to be a q'-group, it is easy to verify that $O_q(G) = 1$. Thus $\text{Fit}(G) = 1$.

(3.6) Example: We present irreducible systems in which $[L : A]$ is not a power of a prime. Let K be any finite non-abelian simple group and let K_p be a nontrivial Sylow p-subgroup of K. Choose a finite nontrivial group R and define $G = K \text{wr} R$ in a way similar to Example (3.5). Let $A = K_p \times K \times \cdots \times K < G$ ($|R| - 1$ copies of K), then $\Gamma_{(A,G)}$ is disconnected (for details see [K], section 3). We have $H = K \times \cdots \times K$ ($|R|$ copies). Similarly to Example (3.5), we obtain (by the simplicity of K and the transitivity of R) that the system $A < H < G$ is irreducible. Furthermore $L = H$, and so $[L : A] = [K : K_p]$. Now since K is nonsolvable, it follows by a theorem of Burnside that $|K|$ is divisible by at least three primes. Thus $[L : A]$ is not a power of a prime. By Corollary A$'$(ii) we have $\text{Fit}(G) = 1$.

4. General systems

We begin this section with the following

Remark: For any system $A < H < G$, A is not a finite nilpotent group. Indeed, suppose on the contrary that A is finite and nilpotent. Let p and q be primes such that p divides a certain number in D_1 and q divides a certain number in D_2. Then for each $g \in G$, pq does not divide $[A : A \cap A^g]$. Let A_p and A_q be the respective Sylow subgroups of A. Then for each $g \in G$ either $A^g > A_p$ or $A^g > A_q$ holds. This implies either $g \in N_G(A_p)$ or $g \in N_G(A_q)$, and so $G = N_G(A_p) \cup N_G(A_q)$. Thus either $G = N_G(A_p)$ or $G = N_G(A_q)$, which contradicts our assumption that both p and q divide some subdegrees. ∎

We have already mentioned that if G satisfies the maximal condition on normal subgroups and $A < H < G$ is a system, then there exists N such that $AN/N < H/N < G/N$ is an irreducible system. The following proposition describes how to find such N.

(4.1) PROPOSITION: *Let a group G possess a system $A < H < G$. Suppose that N is a subgroup of G, which is maximal such that* (i) $N \lhd G$ *and* (ii) $AN \not\geq L$ *(if G satisfies the maximal condition on normal subgroups then such N certainly exists). Then $AN < H$ and $AN/N < H/N < G/N$ is an irreducible system.*

Proof: Like in the proof of (3.2), we use the notation $g^* = gN$, $M^* = MN/N$. We must prove that $A^* < H^* < G^*$ is an irreducible system. Notice first that $AN < H$ and $A^* < H^* < G^*$ is a system, by (3.2). Furthermore, $L^* = (A^*)^{H^*}$, which shows that L^* is in fact the "new L" of the system $A^* < H^* < G^*$. In view of (3.3), it suffices to show that $A^*K^* \geq L^*$ for each K^* such that $1 < K^* \lhd G^*$. Indeed, let K^* satisfy $1 < K^* \lhd G^*$ and let K be the inverse image of K^*. We have $N < K \lhd G$, whence by our assumption $AK \geq L$. This implies $A^*K^* \geq L^*$, as required. ∎

As a direct result of Proposition (4.1) we obtain the following result on finite groups possessing a system.

(4.2) COROLLARY: *Let a finite group G possess a system $A < H < G$. Then G has a proper normal subgroup N, such that AN/N is not nilpotent and* $\mathrm{Fit}(G/N)$ *is a (possibly trivial) q-group for a certain prime q dividing* $[L : A]$.

Proof: Choose N as described in (4.1). By (4.1) $AN/N < H/N < G/N$ is an irreducible system. According to the preceding remark, AN/N is not nilpotent. By Corollary A$'$(ii), $\mathrm{Fit}(G/N)$ is a q-group for a certain prime q. Furthermore, suppose $\mathrm{Fit}(G/N) \neq 1$. In the proof of (4.1) we have shown that LN/N is the "new L" of the system $AN/N < H/N < G/N$. It follows by Corollary A$'$(ii) that $[LN/N : AN/N]$ is a power of q. Hence q divides $[L : A]$. ∎

Our next goal is to prove Theorem B. In the following we always assume that A is a non-normal subgroup of G such that all the subdegrees of (A, G) are finite.

In Lemmas (4.3) and (4.4) below we do not assume that the group G possesses a system. We say that a pair (A, G) is of type (*) if $[A^G : A]$ is finite and coprime to every subdegree of (A, G).

For example, let $A < H < G$ be a system and consider the pair (A, H). Since the subdegree set of (A, H) is D_1 and $[L : A]$ divides $\gcd(D_2)$ (see Theorem B(ii) in [K]), it holds that (A, H) is of type (*).

The following simple lemma is useful.

(4.3) LEMMA: *Assume that (A, G) is of type (*). Then for each $g \in G - N_G(A)$, AA^g is not a subgroup of G.*

Proof: Let $g \in G$ and suppose that AA^g is a subgroup of G. Thus $AA^g \leq A^G$, and so $[AA^g : A]$ divides $[A^G : A]$. Now $[AA^g : A] = [Ag^{-1}A : A] = [A : A \cap A^{g^{-1}}]$, a subdegree of (A, G). Since (A, G) is of type (*) we obtain $[AA^g : A] = 1$. Consequently, $A^g \leq A$.

Now we have $A \leq A^{g^{-1}}$, so $AA^{g^{-1}}$ is a subgroup of G. Applying the argument of the previous paragraph to $AA^{g^{-1}}$, we get $A^{g^{-1}} \leq A$. Thus $A^{g^{-1}} = A$, which implies $g \in N_G(A)$. ∎

As a corollary of Lemma (4.3) we obtain:

(4.4) LEMMA: *Assume that (A, G) is of type (*).*
(i) *Let M satisfy $M \geq A$ and $M \cap A^G \leq N_G(A)$. Then $M \leq N_G(A)$.*
(ii) *The normal closure A^G is not contained in $N_G(A)$.*
(iii) *$N_G(N_G(A)) = N_G(A)$.*

Proof: (i) Let $g \in M$, then $A^g \leq M \cap A^G \leq N_G(A)$, and so AA^g is a subgroup of G. Thus $g \in N_G(A)$ by (4.3).

(ii) Suppose on the contrary that $A^G \leq N_G(A)$. Thus $G \cap A^G = A^G \leq N_G(A)$, whence by (i) we have $G \leq N_G(A)$. This contradicts our assumption that A is a non-normal subgroup of G.

(iii) Let $g \in N_G(N_G(A))$, then $A^g \leq N_G(A)$, and so AA^g is a subgroup of G. Thus $g \in N_G(A)$ by (4.3). ∎

The following proposition about systems will be helpful in proving Theorem B.

(4.5) PROPOSITION: *Let a group G possess a system $A < H < G$. Then*
(i) *A is not normal in L.*
(ii) *If M is a subgroup of G such that $M \geq A$ and $M \cap L \leq N_G(A)$, then $M \leq N_G(A)$.*

Proof: (i) Apply (4.4)(ii) to the pair (A, H).

(ii) Suppose that $M \geq A$ and $M \cap L \leq N_G(A)$. From $M \geq A$ it follows that either $M \geq L$ or $M \leq H$ (see Theorem B(iii) of [K]). If $M \geq L$ then $L \leq N_G(A)$, contradicting (i). Hence $M \leq H$, and the proof is concluded by applying (4.4)(i) to the pair (A, H). Notice that $N_G(A) = N_H(A)$ by Theorem (1.1). ∎

We are ready now for

Proof of Theorem B: Notice again that $N_G(A) = N_H(A)$ by Theorem (1.1). By applying (4.4)(iii) to the pair (A, H), we obtain $N_H(N_G(A)) = N_G(A)$. Thus $N_G(N_G(A)) \cap L \leq N_G(A)$. Now from (4.5)(ii) it follows that $N_G(N_G(A)) = N_G(A)$. ∎

Let $A < H < G$ be a system. We remark that A (unlike $N_G(A)$) does not have to be a self-normalizer. In fact, let G_0 be a direct product of G and a nontrivial group M. Then it is easily verified that $A < HM < G_0$ is a system and $N_{G_0}(A) = N_G(A)M > A$.

We conclude this section by the following proposition on systems, which describes an interesting property of the intersections $H \cap AA^g$ (where $g \in G$).

(4.6) Proposition: *Let a group G possess a system $A < H < G$. Then*

(i) *For every $u \in H - N_G(A)$, AA^u is not a subgroup of H.*

(ii) *For every $v \in G - H$, $H \cap AA^v$ is a subgroup of H. Moreover $H \cap AA^v \geq L$.*

Proof: (i) Just apply (4.3) to the pair (A, H).

(ii) Consider the action of H on the set $\{AgA \mid AgA \subseteq G - H\}$, given by $(AgA)^u = AgAu = AguA$ (see Theorem B(iv) of [K]). For a fixed $v \in G \setminus H$, what is the stabilizer K of AvA with respect to this action? Let $u \in H$, then $u \in K$ iff $AvuA = AvA$ iff $vu \in AvA$ iff $u \in A^vA$. Thus $K = H \cap A^vA = (H \cap A^v)A = A(H \cap A^v) = H \cap AA^v$. Moreover, $AA^v = LL^v$ (by Theorem B(i) of [K]), and so $H \cap AA^v \geq L$.

5. Generalizations of the $\{1, p, q\}$-Theorem

The following claims are needed for the proofs of Theorem C and Corollary C′.

(5.1) Lemma: *Assume that (A, G) is of type (*) and denote $K = A^G$. Suppose further that $[K : A] = q$, a prime. Then $A_K = A_G$.*

Proof: Suppose on the contrary that $A_K \neq A_G$, then there exists $g \in G - N_G(A)$ such that $A_K \not\leq A^g$. Thus $1 \neq [A_K : A_K \cap A^g] = [A^gA_K : A^g] = [A(A_K)^{g^{-1}} : A]$. Now since $K \trianglelefteq G$ and $A_K \triangleleft K$ we have $(A_K)^{g^{-1}} \triangleleft K$. Consequently $A(A_K)^{g^{-1}}$

is a subgroup of K. Hence $[A(A_K)^{g^{-1}} : A]$ divides $[K : A] = q$, which implies $[A(A_K)^{g^{-1}} : A] = q$ and $A(A_K)^{g^{-1}} = K$. It follows that $AA^{g^{-1}} = K$, which contradicts (4.3). ∎

(5.2) LEMMA: *Let a group G possess a system $A < H < G$.*

(i) *Let $v \in G - H$. Then $L^v \leq L$ iff $AA^v = L$ iff $[Av^{-1}A : A] = [L : A]$.*

(ii) *$N_G(L) = H \cup \{v \in G - H \mid [AvA : A] = [Av^{-1}A : A] = [L : A]\}$. Note that the second component in this union may be empty.*

(iii) *$L \lhd G$ iff $D_2 = \{[L : A]\}$.*

Proof: (i) We have $AA^v = LL^v$ (see Theorem B(i) in [K]), whence $L^v \leq L$ iff $AA^v = L$. Moreover, since $AA^v = LL^v$, the set AA^v contains L. Thus $AA^v = L$ iff $[AA^v : A] = [L : A]$. But $[AA^v : A] = [Av^{-1}A : A]$, which concludes the proof of (i).

(ii) Let $g \in G$; then $g \in N_G(L)$ iff $L^g \leq L$ and $L^{g^{-1}} \leq L$. Now fix $v \in G - H$. It follows by (i) that $v \in N_G(L)$ iff $[AvA : A] = [Av^{-1}A : A] = [L : A]$. Since $H \leq N_G(L)$, the result follows.

(iii) This follows from (ii). Recall that $[AgA : A] = [A : A \cap A^g]$ for each $g \in G$. ∎

Theorem C and Corollary C′ can be proved now.

Proof of Theorem C: By applying (5.1) to the pair (A, H) we obtain $A_L = A_H$, whence the quotient H/A_L is defined. For each element $u \in H$ and subgroup $M \leq H$ denote $u^* = uA_L$, an element of H/A_L, and $M^* = MA_L/A_L$, a subgroup of H/A_L. We have $A^* \neq 1$ (by Proposition (4.5)(i)) and $[L^* : A^*] = q$.

When considering the standard action of L on $A \setminus L$, we see that L^* is isomorphic to a transitive nonregular permutation group of degree q. Suppose first that L^* is solvable; then by (11.6) in [W] L^* is Frobenius with a complement A^*. Let $r = |A^*|$. We have $L^* \trianglelefteq H^*$, and all the Hall q'-subgroups of L^* (i.e., the complements of L^*) are conjugate in L^*. Hence we may apply Frattini's argument to get $H^* = N_{H^*}(A^*)L^*$. Let $s \in D_1$; then there exists $u \in H$ such that $s = [A : A \cap A^u]$, and there exist $n^* \in N_{H^*}(A^*)$, $l^* \in L^*$ such that $u^* = n^*l^*$. Thus $s = [A : A \cap A^u] = [A^* : A^* \cap (A^*)^{u^*}] = [A^* : A^* \cap (A^*)^{n^*l^*}] = [A^* : A^* \cap (A^*)^{l^*}]$. Since A^* is a Frobenius complement in L^*, we have either $(A^*)^{l^*} = A^*$ or $A^* \cap (A^*)^{l^*} = 1$. But $s \neq 1$, whence $A^* \cap (A^*)^{l^*} = 1$ and so $s = |A^*| = r$. It follows that $D_1 = \{r\}$, and case (a) is obtained.

Suppose now that L^* is nonsolvable; then by (11.7) in [W] L acts 2-transitively on $A \setminus L$, and so L^* is isomorphic to a 2-transitive permutation group of degree

q. Thus for each $u \in L - A$ we have a disjoint union $L = A \cup AuA$, whence $q - 1 = [L : A] - 1 = [AuA : A] \in D_1$. This provides case (b).

Finally notice that in both cases $[A : A_L]$ divides $(q-1)!$, since $[L : A] = q$. Consequently, if A is finite then A_L contains all the Sylow q-subgroups of A. Notice that the Sylow q-subgroups of A are nontrivial, since q divides $\gcd(D_2)$ by Theorem B(ii) of [K]. ∎

Proof of Corollary C′: By Theorem B(ii) of [K] $[L : A] = q$, whence Theorem C applies. Moreover $L \lhd G$ by (5.2)(iii), and since $L = A^H \leq A^G$, we obtain $L = A^G$.

Remark: Let N be the normal subgroup mentioned in the $\{1, p, q\}$-Theorem. By Corollary C′ we have $L = A^G \leq N$ and $[L : A] = q = [N : A]$. Thus the equality $N = L$ is obtained.

In Example (3.4) we presented finite solvable groups G possessing a system $A < H < G$ such that $[L : A] = q$, a prime. This provides an example for case (a) of Theorem C. Also the assumption $D_2 = \{q\}$ of Corollary C′ holds there.

Consider now Example (3.5). Here $L = H = K \times \cdots \times K$ ($|R|$ copies of $K = S_q$), $[L : A] = q$, a prime, and $A_L = 1 \times K \times \cdots \times K$ ($|R| - 1$ copies of K). Thus L/A_L is isomorphic to $K = S_q$, which is nonsolvable (recall that $q \geq 5$), and so case (b) of Theorem C holds. Notice that $D = \{1, q-1, q\}$, so $D_1 = \{q-1\}$ and $D_2 = \{q\}$.

An example for case (b) of Theorem C in which D_1 is not a singleton is given in [P] (see [P], Example (2.2)(b)). In that example $D_1 = \{3, 4, 6\}$ and $D_2 = \{7\}$. Notice further that it follows that the solvability assumption in Theorem D of [K] cannot be omitted.

In the following theorem we add to the setting of Theorem C the assumption that q is a self-paired subdegree of (A, G). Notice that the examples just described satisfy this condition.

THEOREM D: *Let a group G possess a system $A < H < G$. Suppose that there exists $v \in G - H$ such that $[A : A \cap A^v] = [A : A \cap A^{v^{-1}}] = q$, a prime. Then $[L : A] = q$, and so one of the cases* (a) *and* (b) *of* Theorem C *holds. In addition,*

(i) *$A \cap (A_L)^v \lhd A$, and $A/A \cap (A_L)^v$ is isomorphic to L/A_L.*

(ii) *Denote $R = A/A \cap (A_L)^v$, $W = (A \cap A^v)/(A \cap (A_L)^v)$. Then in case* (a) *$W$ is a complement of the Frobenius group R, and in case* (b) *the standard action of R on $W \setminus R$ is 2-transitive and faithful.*

Proof: Notice first that $\gcd(D_2) = q$, whence the condition $[L : A] = q$ of Theorem C is satisfied. Thus one of the cases (a) and (b) of Theorem C holds.

Now we claim that the indices $[A : A\cap A^{v^{-1}}]$ and $[A : A_L]$ are coprime. Indeed, $[A : A \cap A^{v^{-1}}] = q$, while $[A : A_L]$ divides $(q-1)!$, since $[L : A] = q$. It follows that $A = A_L(A \cap A^{v^{-1}})$. Furthermore, since

$$[AvA : A] = [A : A \cap A^v] = [L : A],$$

Lemma (5.2)(i) implies $L = AA^{v^{-1}}$. Consequently

$$L = AA^{v^{-1}} = A_L(A \cap A^{v^{-1}})A^{v^{-1}} = A_L A^{v^{-1}}.$$

We have

$$L/A_L = A^{v^{-1}} A_L/A_L,$$

and $A^{v^{-1}} A_L/A_L$ is isomorphic to $A^{v^{-1}}/A^{v^{-1}} \cap A_L$. This last quotient is clearly isomorphic to $A/A \cap (A_L)^v$, and so we have part (i).

For proving (ii), let ϕ denote the natural isomorphism from L/A_L to $A^{v^{-1}}/A^{v^{-1}} \cap A_L$ (recall that $L/A_L = A^{v^{-1}} A_L/A_L$), and let ψ denote the obvious isomorphism from $A^{v^{-1}}/A^{v^{-1}} \cap A_L$ to $A/A \cap (A_L)^v$. We have $(L/A_L)^{\phi\psi} = A/A \cap (A_L)^v = R$. Furthermore, $A = A_L(A \cap A^{v^{-1}})$, and so $(A/A_L)^\phi = ((A \cap A^{v^{-1}})A_L/A_L)^\phi = (A \cap A^{v^{-1}})/(A^{v^{-1}} \cap A_L)$. Thus

$$(A/A_L)^{\phi\psi} = (A \cap A^v)/(A \cap (A_L)^v) = W.$$

This implies part (ii), since one of the cases (a) and (b) of Theorem C holds (in the proof of Theorem C we have noted that in case (b) L acts 2-transitively on $A \setminus L$). ∎

Let A and K be groups such that A acts on K via automorphisms, and set $G = AK$, the respective semidirect product. It is easily checked (see [K], the first paragraph of section 4) that the subdegrees of (A, G) are exactly the cardinalities of the A-orbits on K. More precisely, for each $k \in K$ we have $[A : A\cap A^k] = |k^A|$, where k^A denotes the A-orbit of k. Suppose that the action of A is nontrivial and that all the A-orbits are finite. In this case the common divisor graph Γ of (A, G) is defined: its vertices are the sizes of the nonsingleton A-orbits, and two different vertices are joined by an edge iff the respective orbit sizes are not coprime. It turns out that in this case every subdegree is paired only with itself, so if Γ is disconnected then a system $A < H < G$ occurs. Indeed, let $g \in G$, $g = ak$, where $a \in A$, $k \in K$. Then $[A : A\cap A^{g^{-1}}] = [A : A\cap A^{k^{-1}a^{-1}}] = [A : A\cap A^{k^{-1}}] = |(k^{-1})^A| = |\{(k^{-1})^a | \ a \in A\}| = |\{(k^a)^{-1} \mid a \in A\}| = |k^A| = [A : A \cap A^k] = [A : A \cap A^{ak}] = [A : A \cap A^g]$.

We shall call Γ the common divisor graph **related to the action of A on K**. Our remark and Theorem D imply

THEOREM E: *Let a group A act nontrivially on a group K via automorphisms, and suppose that all the A-orbits are finite. Assume that the common divisor graph Γ related to this action is disconnected, and there exists a prime q such that $q \in D_2$. Then A has a normal subgroup N such that one of the following cases holds:*

(a) *A/N is Frobenius with a kernel of order q and a complement of order, say, r. In this case $D_1 = \{r\}$.*

(b) *A/N is isomorphic to a nonsolvable 2-transitive permutation group of degree q. In this case $q-1 \in D_1$.*

We do not know whether there exists a system $A < H < G$ such that G is a simple group. However, the following proposition asserts that if such a system exists, then D_2 must be a relatively "rich" set.

(5.3) PROPOSITION: *Assume that Γ is disconnected and $D_2 \subseteq \{q, q^2\}$, where q is a prime (since D_2 is finite, A must be stable in G under these conditions, whence we have a system $A < H < G$). Then G is not simple.*

Proof: If $D_2 = \{q\}$ then $L \lhd G$ by Corollary C$'$. Suppose then that $q^2 \in D_2$. By Theorem B(ii) of [K], either $[L : A] = q^2$ or $[L : A] = q$ holds. If $[L : A] = q^2$ then (by the same theorem) $D_2 = \{q^2\}$. Hence (5.2)(iii) implies $L \lhd G$ and G is not simple. Let $[L : A] = q$; then (see Theorem C in [K]) the subdegree set of (L, G) is $\{1, q\}$. Thus Theorem 2 in [BL] implies the existence of a subgroup N, $N \unlhd G$, such that either (a) $L \lhd N$ and $[N : L] = q$, or (b) $N \lhd L$ and L/N is isomorphic to a transitive permutation group of degree q.

Suppose that (a) holds. Since $[L : A] = q$ and $q^2 \in D_2$, it follows by (5.2)(iii) that L is not normal in G. Hence N is a proper normal subgroup of G. Suppose now that (b) holds; then $[L : N]$ divides $q!$, whence q^2 does not divide $[L : N]$. But since $q^2 \in D_2$, we have $[A : A \cap A^v] = q^2$ for some $v \in G - H$. It follows that N is nontrivial, which concludes the proof. ∎

References

[BL] G. M. Bergman and H. W. Lenstra, Jr., *Subgroups close to normal subgroups*, Journal of Algebra **127** (1989), 80–97.

[IP] I. M. Isaacs and C. E. Praeger, *Permutation group subdegrees and the common divisor graph*, Journal of Algebra **159** (1993), 158–175.

[K] G. Kaplan, *On groups admitting a disconnected common divisor graph*, Journal of Algebra **193** (1997), 616–628.

[P] C. E. Praeger, *Subgroups close to normal subgroups: a commentary on a paper of G.M. Bergman and H.W. Lenstra, Jr.*, Mathematics Research Report, Australian National University, 1989.

[W] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.